PMM U.S.S.R.,Vol.50,No.4,pp.433-437,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

ON THE MOTION OF A CONTROLLED SYSTEM OF VARIABLE MASS*

A.G. AZIZOV

A mechanical system with servoconstraints whose motion is controlled by reactive forces is investigated. The law of variation of mass of the system ensuring the realization of the servoconstraints is determined, and the problem of stabilizing the motion with respect to a manifold defined by these constraints is studied. The method of investigation is based on the rules of combination of the constraints /1/ and the Chetayev's theory of parametric release /2/.

The general theory of motion of a system with non-ideal constraints applied to problems with friction /1, 3/ was used in /4/ to construct the equations of motion of a control system with constraints whose response were reactive forces. However, the systems /4/ in which the laws of variation of mass were known in advance, and all constraints effected by reactive forces were applied exactly over the whole period of motion, embrace only a narrow class of problems. A more general case is of interest, when only a part of the constrains rely on reactive forces, where the possible deviations of the motions from the servoconstraints are taken into account and the laws governing the variation of mass of the points are not known in advance and are found from the differential equations supplementing the equations of motion of the system.

1. We consider a mechanical system of material points M_k $(k = 1, 2, \ldots, n)$ whose positions in the inertial frame of reference are given by their Cartesian coordinates x_ν $(\nu = 1, 2, \ldots, 3n)$. Let the given forces $\mathbf{F}_k$ (X_ν) belonging to class C_1 act on the points, and let their motion be constrained by the compatible and independent constraints which include the geometrical constraints

$$f_\alpha (x_\nu, t) = 0, \ (f_\alpha \in C_2; \ \alpha = 1, 2, \ldots, a) \tag{1.1}$$

as well as the kinematic constraints which are, in general, non-linear

$$\varphi_\beta (x_\nu, x_\nu^{\cdot}, t) = 0 \quad (\varphi_\beta \in C_1; \ \beta = 1, 2, \ldots, b) \tag{1.2}$$

The possible displacements allowed by the constraints will be determined by $a + b$ independent relations /2/

$$\sum_{\nu=1}^{3n} \frac{\partial f_\alpha}{\partial x_\nu} \delta x_\nu = 0, \quad \sum_{\nu=1}^{3n} \frac{\partial \varphi_\beta}{\partial x_\nu^{\cdot}} \delta x_\nu = 0$$

and the manifold of admissible states of the system will be represented in the form

$$x_\nu = a_\nu (q_i, t), \quad x_\nu^{\cdot} = b_\nu (q_i, p_j, t) \, (a_\nu \in C_2, \ b_\nu \in C_1) \tag{1.3}$$

where q_i $(i = 1, 2, \ldots, p)$ are independent Lagrangian coordinates and p_j $(j = 1, 2, \ldots, r)$ are independent velocity parameters. The variations in the Cartesian coordinates can be expressed in terms of arbitrary quantities $\delta \pi_j$ as follows:

$$\delta x_\nu = \sum_{j=1}^{r} \frac{\partial b_\nu}{\partial p_j} \delta \pi_j$$

We shall assume that the constraints will be divided, according to the method of their implementation, into constraints of the first kind /5/ and servoconstraints whose responses will be automatically regulated reactive forces produced by the points of the system. Let the first c constraints of (1.1) and the first d constraints of (1.2) be constraints of the first kind. Denoting by $\mathbf{N}_k$ (N_ν) the reaction forces of the constraints of the first kind and by $\mathbf{\Phi}_k$ (Φ_ν) the reaction forces of the servoconstraints, we write the resulting responses $\mathbf{R}_k$ (R_ν) as $R_\nu = N_\nu + \Phi_\nu$. Here the axiom of ideal constraints will be represented by the equation

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*Prikl.Matem.Mekhan.,50,4,567-572,1986

$$\sum_{\nu=1}^{3n} R_\nu \delta x_\nu = 0$$

valid for any possible displacements. The necessary and sufficient condition of this validity will be, that the condition /3/

$$R_\nu = \sum_{\alpha=1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu} + \sum_{\beta=1}^{b} \mu_\beta \frac{\partial \varphi_\beta}{\partial x_\nu^{\cdot}}$$

holds, where λ_α and μ_β are the undetermined Lagrange multipliers.

We will assume that the sum of elementary works by the constraint reaction forces over any possible displacement will be

$$\sum_{\nu=1}^{3n} R_\nu \delta x_\nu = \tau \neq 0 \tag{1.4}$$

and the constraints of the first kind are ideal. In this case the servoconstraints will not be ideal and it will be possible to use the rule of combined constraints to study the system.

Indeed, taking into account initially only the servoconstraints of the system, we will write the expression for the manifold of admissible states in the form

$$\begin{gathered} x_\nu = A_\nu(q_j, t), \quad x_\nu^{\cdot} = B_\nu(q_j, p_\sigma, t) \\ (A_\nu \in C_2,\ B_\nu \in C_1;\ j = 1, 2, \ldots, m = p + c;\ \sigma = 1, 2, \ldots, l = r + d) \end{gathered} \tag{1.5}$$

Under the above assumptions concerning the ideal nature of the constraints of the first kind, we obtain the following expression from (1.4):

$$\sum_{\nu=1}^{3n} \Phi_\nu \delta x_\nu = \tau \tag{1.6}$$

valid for any possible displacement, and the servoconstraint reaction force $\mathbf{\Phi}_k$ can be expanded in a unique manner into the components $\mathbf{\Phi}_k^n$ and $\mathbf{\Phi}_k^\tau$ such, that the left-hand side is equal to zero for Φ_ν^n and the vectors $\mathbf{\Phi}_k^\tau\, \delta t$ appear amongst the possible displacements. Moreover, we have

$$\Phi_\nu^n = \sum_{\alpha=c+1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu} + \sum_{\beta=d+1}^{b} \mu_\beta \frac{\partial \varphi_\beta}{\partial x_\nu^{\cdot}}$$

$$\Phi_\nu^\tau = \sum_{\sigma=1}^{l} u_\sigma \frac{\partial B_\nu}{\partial p_\sigma} \quad (u_\sigma \in C_1)$$

where u_σ are certain coefficients of proportionality.

The motion of the points of the system will be described by the equations

$$m_\nu x_\nu^{\cdot\cdot} = X_\nu + \sum_{\alpha=1}^{a} \lambda_\alpha \frac{\partial f_\alpha}{\partial x_\nu} + \sum_{\beta=1}^{b} \mu_\beta \frac{\partial \varphi_\beta}{\partial x_\nu^{\cdot}} + \Phi_\nu^\tau \tag{1.7}$$

and these should be supplemented by the constraint Eqs.(1.1) and (1.2) and the mass variation equations.

We will derive the differential equations of the variation of mass by turning to the expression for the reaction force $m_k^{\cdot} V_k^r$ $(k = 1, 2, \ldots, n)$ produced by the point M_k of the system where $m_k^{\cdot}$ is the rate of loss of mass per second and $\mathbf{V}_k^r$ is the relative velocity of the ejected particles. The following relations hold:

$$m_k^{\cdot} \mathbf{V}_k^r = \mathbf{\Phi}_k \quad (k = 1, 2, \ldots, n)$$

and they yield the following system of differential equations for determining the rate of loss of mass of the points:

$$m_k^{\cdot} V_k^r = -\sqrt{\mathbf{\Phi}_k^2} \quad (k = 1, 2, \ldots, n) \tag{1.8}$$

where V_k^r is the relative velocity of the ejected particles.

Differentiating Eqs.(1.1) with respect to time twice and (1.2) once, and replacing $x_\nu^{\cdot\cdot}$ by their values from (1.7), we obtain $a + b$ linear equations which enable us to determine the multipliers λ_α and μ_β as functions of the coordinates x_ν, the velocities $x_\nu^{\cdot}$, the masses m_k, the time t and the arbitrary parameters u_σ $(\sigma = 1, 2, \ldots, l)$. Consequently we obtain $3n$ second-order Eqs.(1.7) and n first-order Eqs.(1.8) containing the parameters $u_1, u_2, \ldots, u_l$ as the controlled quantities, for determining the motion of the points of the system and the law of variation of mass.

2. As we know /6/, the servoconstraints are represented by the invariant relations of the differential equations of motions obtained. When perturbations appear, violating the conditions of servoconstraints, the question arises concerning the need to take into account the fact that the system can be freed, and the solution of the problem on stabilizing the motions relative to the manifold determined by the servoconstraints. Taking into account this formulation of the problem and the equations of servoconstraints from systems (1.1) and (1.2), we will also consider the equations

$$f_{c+\gamma}(x_\nu, t) = \eta_\gamma \quad (\gamma = 1, 2, \ldots, e = a - c)$$
$$\varphi_{d+\rho}(x_\nu, x_\nu^{\cdot}, t) = \zeta_\rho \quad (\rho = 1, 2, \ldots, f = b - d) \tag{2.1}$$

where η_γ and ζ_ρ are parameters characterizing the continuous release of the system from the geometrical and kinematic constraints. In the case of such a parametric release, we take the left-hand sides of the equations of servoconstraints, which can be calculated for the real motion /7/, as the deviations, and in place of (1.5) we can obtain the following expressions for the manifold of admissible states:

$$x_\nu = A_\nu^*(q_j, \eta_\gamma, t) \quad (A_\nu^* \in C_2)$$
$$x_\nu^{\cdot} = B_\nu^*(q_j, \eta_\gamma, \zeta_\rho, p_\sigma, \eta_\gamma^{\cdot}, t) \quad (B_\nu^* \in C_1) \tag{2.2}$$

When $\eta_\gamma = \zeta_\rho = \eta_\gamma^{\cdot} = 0$, we add Eqs.(1.5), determining the manifold of admissible states of the system which has not been freed.

Adopting for Eqs.(2.1) the definition of possible displacements for systems with parametric constraints /6/, we obtain the conditions

$$\sum_{\nu=1}^{3n} \frac{\partial f_{c+\gamma}}{\partial x_\nu} \delta x_\nu = 0, \quad \sum_{\nu=1}^{3n} \frac{\partial \varphi_{d+\rho}}{\partial x_\nu^{\cdot}} \delta x_\nu = 0$$

enabling us to represent the variations in Cartesian coordinates in terms of the arbitrary quantities $\delta\pi_\sigma$ as follows:

$$\delta x_\nu = \sum_{\sigma=1}^{l} \frac{\partial B_\nu^*}{\partial p_\sigma} \delta\pi_\sigma$$

Considering expression (1.4) and assuming that $c + d$ constraints of the first kind are ideal, we obtain Eq.(1.6) which holds for any possible displacement. We can decompose, as before, the force of reaction of the servoconstraints Φ_k into the components Φ_k^n and Φ_k^τ, and

$$\Phi_\nu^\tau = \sum_{\sigma=1}^{l} u_\sigma \frac{\partial B_\nu^*}{\partial p_\sigma} \tag{2.3}$$

Replacing Φ_ν^τ in (1.7) and (1.8) by their values from (2.3) and supplementing them with the equations of constraints, we obtain the multipliers λ_α and μ_β as functions of the coordinates x_ν, the velocities $x_\nu^{\cdot}$, masses m_k, time t and the arbitrary parameters u_σ $(\sigma = 1, 2, \ldots, l)$, and of the release parameters η_γ and ζ_ρ and their derivatives $\eta_\gamma^{\cdot}, \zeta_\rho^{\cdot}, \eta_\gamma^{\cdot\cdot}$.

Introducing the notation

$$\eta_\gamma^{\cdot} = y_\gamma, \quad \zeta_\rho = y_{e+\rho}, \quad \eta_\gamma = y_{q+\gamma}$$
$$\eta_\gamma^{\cdot\cdot} = V_\gamma, \quad \zeta_\rho^{\cdot} = V_{e+\rho} \quad (q = e + f)$$

we obtain the following system of equations:

$$y^{\cdot} = Ay + BV \tag{2.4}$$
$$y = \begin{Vmatrix} y_1 \\ y_2 \\ \vdots \\ y_\pi \end{Vmatrix}, \quad A = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ E_f & 0 & 0 \end{Vmatrix}, \quad B = \begin{Vmatrix} E_f & 0 \\ 0 & E_e \\ 0 & 0 \end{Vmatrix}, \quad V = \begin{Vmatrix} V_1 \\ V_2 \\ \vdots \\ V_q \end{Vmatrix} (\pi = 2e + f)$$

where E_f and E_e are unit submatrices of order $f \times f$ and $e \times e$ respectively.

System (2.4), which describes the deviation of the motion from the servoconstraints, is fully controllable /8/ and an equation of the form $V = V(y)$ $(V(0) = 0)$ can always be found for it, ensuring the stabilization of the zeroth-order solution of the equations

$$y^{\cdot} = Ay + BV(y), \ y(0) = y^0 \tag{2.5}$$

Considering now the equations of motion of the system together with the equations of variation of mass (1.8), associated equations of the contraints of the first kind from systems (1.1) and (1.2) and Eqs.(2.5), and taking into account (2.1), we obtain $3n$ equations of the second order in the coordinates, and n equations of the first order in the masses.

3. When constructing Eqs.(2.2), we disregarded c geometrical and d kinematic constraints of the first kind from systems (1.1) and (1.2). In order to include these constraints, we transform them to the variables defining the manifold (2.2) and assume that the resulting geometrical constraints can be solved for the variables $q_{p+1}, \ldots, q_m$ and the kinematic constraints for the variables $p_{r+1}, \ldots, p_e$.

We obtain the following equations for the manifold of admissible states of such a system:

$$x_\nu = a_\nu^*(q_i, \eta_\gamma, t) \quad (a_\nu^* \in C_2;\ \ i = 1, 2, \ldots, p) \tag{3.1}$$
$$x_\nu^{\cdot} = b_\nu^*(q_i, \eta_\gamma, \zeta_\rho, p_j, \eta_\gamma^{\cdot}, t) \quad (b_\nu^* \in C_1;\ \ j = 1, 2, \ldots, r)$$

and the variations in Cartesian coordinates will be expressed in terms of the ordinary quantities $\delta\pi_j$ as follows:

$$\delta x_\nu = \sum_{j=1}^{r} \frac{\partial b_\nu^*}{\partial p_j} \delta\pi_j$$

Let us multiply each equation of motion of the points of the system by δx_ν, and add them together. Introducing the energy of accelerations of the system S, we will have

$$\sum_{j=1}^{r} \Big[\sum_{\nu=1}^{3n} \Big(\frac{\partial S}{\partial x_\nu^{\cdot\cdot}} - X_\nu - \Phi_\nu^\tau \Big) \frac{\partial b_\nu^*}{\partial p_j} \Big] \delta\pi_j = 0; \quad S = \frac{1}{2} \sum_{\nu=1}^{3n} m_\nu x_\nu^{\cdot\cdot 2} \tag{3.2}$$

Differentiating (3.1) with respect to time, we obtain

$$x_\nu^{\cdot\cdot} = \sum_{j=1}^{r} \frac{\partial b_\nu^*}{\partial p_j} p_j^{\cdot} + \ldots$$

where repeated dots denote terms not containing the derivatives of the velocity parameters p_j. Transforming the expression (3.2) with the help of the identities $\partial x_\nu^{\cdot\cdot}/\partial p_j^{\cdot} = \partial b_\nu^*/\partial p_j$ and taking into account the arbitrariness of the quantities $\delta\pi_j$, we obtain the following system of equations:

$$\frac{\partial S^*}{\partial p_j^{\cdot}} = Q_j^* + \Phi_j^*; \quad Q_j^* = \sum_{\nu=1}^{3n} X_\nu \frac{\partial b_\nu^*}{\partial p_j}, \quad \Phi_\nu^* = \sum_{\nu=1}^{3n} \Phi_\nu^\tau \frac{\partial b_\nu^*}{\partial p_j} \tag{3.3}$$

where S^* is the energy of accelerations constructed using Eqs.(3.1).

If we now pass, in Eqs.(1.8) and (3.3), from the Cartesian coordinates and their derivatives to the variables used to define Eqs.(3.1) and add to them Eqs.(2.5) and the kinematic relations

$$q_i^{\cdot} = q_i^{\cdot}(q_s, \eta_\gamma, \zeta_\rho, \rho_j, \eta_\gamma^{\cdot}, t) \quad (q_i^{\cdot} \in C_1;\ \ i, s = 1, 2, \ldots, p)$$

which occur by virtue of the presence of kinematic contraints (1.2), we obtain a complete system of equations for determining the unknowns m_k, p_j, q_i, y_ξ $(\xi = 1, 2, \ldots, \pi)$. Here Eqs.(1.8) and (3.3) will contain arbitrary parameters $u_1, u_2, \ldots, u_l$.

4. We will consider, as an example, the problem given in /9, Sect.10/, assuming that the points M_1 and M_2 have masses m_1 and m_2 respectively and the non-holonomic constraints reduces to the condition

$$x_1^{\cdot}(y_2 - y_1) - y_1^{\cdot}(x_2 - x_1) = 0 \tag{4.1}$$

which means that the velocity of the point M_1 must be directed along the rod $M_1M_2 = l$. Assuming that the reactive force is produced by the point M_1 only and expression (4.1) corresponds to the servoconstraint, we construct the equations of motion of the system and the equations of variation of mass.

Restricting ourselves initially to the case when the servoconstraint is satisfied exactly by the relations $x_1^{\cdot} = p(x_2 - x_1)$, $y_1^{\cdot} = p(y_2 - y_1)$ which satisfy identically the condition (4.1), we introduce the high speed parameter p and write the force of reaction of the servoconstraint in the form

$$\Phi_{x1} = \mu(y_2 - y_1) + u(x_2 - x_1), \quad \Phi_{y1} = \mu(x_1 - x_2) + u(y_2 - y_1)$$

where μ is the servoconstraint multiplier and u is an arbitrary parameter. Taking into account the geometrical constraint of this problem, regarded as a constraint of the first kind, we will represent the manifold of admissible velocities thus:

$$x_1^{\cdot} = lp\cos\varphi, \quad x_2^{\cdot} = l(p\cos\varphi - \varphi^{\cdot}\sin\varphi) \tag{4.2}$$
$$y_1^{\cdot} = lp\sin\varphi, \quad y_2^{\cdot} = l(p\sin\varphi + \varphi^{\cdot}\cos\varphi)$$

Writing out Eqs.(3.3), we obtain

$$(m_1 + m_2)(p^{\cdot} + k \sin\varphi) - m_2\varphi^2 = u \tag{4.3}$$
$$\varphi^{\cdot\cdot} + p\varphi^{\cdot} + k\cos\varphi = 0 \quad (k = g/l)$$

The value of the multiplier of the servoconstraint will in this case be $\mu = -m_1(p\varphi^{\cdot} + k\cos\varphi)$, therefore using (1.8) we obtain

$$m_1^{\cdot} = -l/V^r\,[m_1^2(p\varphi^{\cdot} + k\cos\varphi)^2 + u^2]^{1/2} \tag{4.4}$$

where V^r is the relative velocity of the ejected particles.

Thus Eqs.(4.3) containing the arbitrary parameter u, describe the motion of the system along the manifold defined by the constraints, and the law of variation of mass governing this motion satisfies Eq.(4.4).

Let us assume that the initial conditions of the system are incompatible with Eq.(4.1) and, that we require to solve the problem of stabilizing motions relative to the manifold in question.

The expressions

$$x_1^{\cdot} = lp\cos\varphi + \frac{\zeta}{2l\sin\varphi}, \quad x_2^{\cdot} = l(p\cos\varphi - \varphi^{\cdot}\sin\varphi) + \frac{\zeta}{2l\sin\varphi}$$
$$y_1^{\cdot} = lp\sin\varphi - \frac{\zeta}{2l\cos\varphi}, \quad y_2^{\cdot} = l(p\sin\varphi + \varphi^{\cdot}\cos\varphi) - \frac{\zeta}{2l\cos\varphi}$$

transformed into Eqs.(4.2) when $\zeta = 0$, satisfy the Chetayev's release algorithm. From the kinematic point of view the parameter ζ represents a quantity characterizing the deviation of the motion of the system from the seroconstraint (4.1).

Writing the equations of motion in the form (3.3) and adding to them the equations

$$\zeta^{\cdot} = V(\zeta), \quad V(0) = 0, \quad \zeta(0) = \zeta^0$$

with an asymptotically stable zero-order solution, we obtain the system

$$(m_1 + m_2)(p^{\cdot} + k\sin\varphi) - m_2\varphi^{\cdot 2} = \tag{4.5}$$
$$u - \frac{m_1 + m_2}{2l^2}[2V(\zeta)\operatorname{ctg}2\varphi + \zeta\varphi^{\cdot}(\operatorname{tg}^2\varphi + \operatorname{ctg}^2\varphi)]$$
$$\varphi^{\cdot\cdot} + p\varphi^{\cdot} + k\cos\varphi = l^{-2}[V(\zeta) + \zeta\varphi^{\cdot}\operatorname{ctg}2\varphi]$$

Supplementing these equations with the equations of variation of mass

$$m_1^{\cdot} = -l/V^r\sqrt{\mu^2 + u^2} \tag{4.6}$$
$$(\mu = m_1\{l^{-2}[V(\zeta) - 2\zeta\varphi^{\cdot}\operatorname{ctg}2\varphi] - p\varphi^{\cdot} - k\cos\varphi\})$$

we obtain the complete system of differential equations of the problem, and Eqs.(4.5) and (4.6) become, as $\zeta \to 0$, (4.3) and (4.4) respectively, defining the motion of the system along the manifold.

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Translated by L.K.